THE KINETIC THEORY FOR DILUTE SOLID/LIQUID TWO-PHASE FLOW

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Abstract—On the basis of a brief review of the continuum theory for macroscopic descriptions and the kinetic theory for microscopic descriptions in solid/liquid two-phase flows, some suggestions are presented, i.e. the solid phase may be described by the Boltzmann equation and the liquid phase still be described by conservation laws in the continuum theory. Among them the action force on the particles by the liquid fluid is a coupling factor which connects the phases. For dilute steady solid/liquid two-phase flows, the particle velocity distribution function can be derived by analogy with the procedures in the kinetic theory of gas molecules for the equilibrium state instead of being assumed, as previous investigators did. This done, more detailed information, such as the velocity probability density distribution, mean velocity distribution and fluctuating intensity etc. can be obtained directly from the particle velocity distribution function or from its integration. Experiments have been performed for dilute solid/liquid two-phase flow in a $4 \times 6 \, \text{cm}^2$ sized circulating square pipe system by means of laser Doppler anemometry so that the theories can be examined. The comparisons show that the theories agree very well with all the measured data.

Key Words: kinetic theory, continuum theory, velocity distribution function

1. INTRODUCTION

A lot of problems are related to two-phase flows, and so various theories have been developed to describe them. Generally speaking, a solid/liquid two-phase flow system can be described either in terms of a macroscopic or microscopic method, or the continuum theory (Soo 1967; Marble 1970; Ishii 1975; Pai 1977; Drew 1983 etc.) or the kinetic theory (Marble 1963; Culic 1964; Pai 1971, 1977; Ishii 1975; Liu 1987; Wang 1989 etc.). Among the cited work using the kinetic approach, Pai's (1971, 1977) was prominent. In his work, special attention was paid to the solid/gas two-phase flows and both the motion of the gas molecules and the solid particles are described by the Boltzmann equation, in which either collisions of similar particles (e.g. those of solid particles or of gas molecules) or those between solid particles and gas molecules are included in the r.h.s. collision term. In addition, the force between solid particles and gas molecules is also included in the force term given on the l.h.s. of the equation. In fact, the interactions between phases should not be included repeatedly. Furthermore, noting that the kinetic theory for microscopic descriptions is primarily used in the study of molecular movement in single gas phase flows or that of gas/solid two-phase flows and the successful applications of the macroscopic method in fluid mechanics, perhaps a worthwhile investigation would be to introduce the kinetic theory into the study of solid/liquid two-phase flow by analogy, i.e. each solid particle can be taken as analogous to a molecule of gas in gas kinetic theory and be described by the Boltzmann equation. This makes us consider the possibility of utilizing the continuum theory for the liquid phase and the kinetic theory for the solid phase in solid/liquid two-phase flows. The reasons are as follows:

(1) The continuum theory, as the term suggests, demands that numerous particles (or molecules) be included in a micelle, which is basically reasonable for the liquid fluid because of its small-sized molecules. For the solid particles in two-phase flow, we have no sound reasoning to use the continuum model if the large dimensions of the particles are considered.

(2) The development of the basic equations in continuum theory is often related to some mean methods. In such a way, additional unknown correlation terms will

be introduced and this will make the problems more difficult to solve, especially for solid/liquid flows.

(3) When the particle concentrations are high, the interactions among the particles become more and more important and they must be considered, but the traditional continuum theory itself can not tell us how the particles affect each other and we have to resort to other theories.

The dilemma mentioned above could be resolved more or less by making us of the microscopic kinetic theory. Firstly, as an initial approximate, the equations of the continuum theory can be derived from the Boltzmann equation of the kinetic theory. Moreover, detailed information on the particle transportation coefficients, which are taken as state parameters in the continuum theory, such as viscosity and heat conduction coefficients etc., can be obtained from the Boltzmann equation just as those given in the kinetic theory of gases. Secondly, a striking advantage of the kinetic theory is that fewer unknown parameters will be introduced by this theory, from which more microscopic parameters can be obtained. For example, parameters such as the probability density distribution of particle velocity are usually not involved in the continuum theory, but such parameters can be readily obtained from the Boltzmann equation. Thirdly, solid particles with any physical properties can be included because the object of study in the kinetic theory is solely the kinetic characteristics of individual particles. Finally, a problem involved by the kinetic theory, the action of collisions which becomes a principal factor for two-phase flow with high particle concentrations, is dealt with by the Boltzmann equation.

Although the kinetic theory has the advantages expounded above, little work (Lourenco et al. 1983; Wang & Ni 1990) has been performed due to the difficulties in the mathematics. Based on the above discussions, a study on dilute solid/liquid two-phase flow with a combination of the kinetic theory and the continuum theory was made as described below.

2. BASIC EQUATIONS OF SOLID/LIQUID TWO-PHASE FLOW

The basic equations of solid/liquid two-phase flow consist of the Boltzmann equation in the kinetic theory for the particles and the conservation equations in the continuum theory for the liquid fluid. Among them the Boltzmann equation is expressed as

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{F}f) = \left(\frac{\partial f}{\partial t}\right)_c, \qquad [1]$$

in which, $f = f(\mathbf{v}, \mathbf{x}, t)$ represents the particle velocity distribution function, which is a function of the particle stochastic velocity $\mathbf{v}$, the spatial coordinate $\mathbf{x}$ and time t; $\mathbf{F}$ represents the unit mass external forces on a particle, including both the gravitational force $\mathbf{g}$ and the action force on the particle by the liquid phase; the r.h.s. of the equation is a collision term reflecting the influences of the collisions between the particles on the velocity distribution function of the particles.

Once the velocity distribution function f is known, all the macroscopic statistical mean parameters for the particle movement will be determined correspondingly, e.g. the particle numbers per unit volume

$$n = \int f \, d\mathbf{v}, \qquad [2]$$

which is connected with the volumetric particle concentration C and the particle phase density by the relations $C = \pi/6D^3 n$ and $\bar{\rho}_P = mn$, respectively. Here D is the particle diameter and m is the mass, the particle mean velocity is expressed as

$$\mathbf{v}_P = \frac{1}{n} \int \mathbf{v} f \, d\mathbf{v}, \qquad [3]$$

then: the particle fluctuating velocity $\mathbf{V}_P$ becomes

$$\mathbf{V}_P = \mathbf{v} - \mathbf{v}_P; \qquad [4]$$

the mean square value of $\mathbf{V}_P$ can be written in the form

$$\overline{V_P^2} = \frac{1}{n} \int (\mathbf{v} - \mathbf{v}_P) \cdot (\mathbf{v} - \mathbf{v}_P) f \, d\mathbf{v}; \qquad [5]$$

and the stress tensor for the particle fluctuation is

$$\mathbf{p}_P = m \int \mathbf{V}_P \mathbf{V}_P f \, d\mathbf{v}; \qquad [6]$$

$\mathbf{p}_P$, in turn, can be divided into two parts which are defined as the pressure stress tensor p_P and the shear stress tensor τ_P, i.e.

$$p_P = \tfrac{1}{3} \int m V_P^2 f \, d\mathbf{v} \qquad [7]$$

and

$$\tau_P = p_P \mathbf{I} - \mathbf{p}_P, \qquad [8]$$

where $V_P^2 = \mathbf{V}_P \cdot \mathbf{V}_P$ and $\mathbf{I}$ is the unit tensor.

In addition, the total external force on the particles per unit volume is

$$\bar{\rho}_P \mathbf{F}_P = m \int \mathbf{F} f \, d\mathbf{v}, \qquad [9]$$

which consists of the total gravitational force $\bar{\rho}_P \mathbf{g}$ per unit volume and the total action force of the liquid fluid, or, the interaction force between the phases:

$$\mathbf{M}_P = m \int \mathbf{F} f \, d\mathbf{v} - \bar{\rho}_P \mathbf{g}. \qquad [10]$$

In the above mentioned formulae, the integral space is $d\mathbf{v} = dv_1 \, dv_2 \, dv_3$, the subscript "P" represents the particle phase.

In view of the successful applications of the continuum theory to the problems of single-phase flow, the liquid fluid can still be described by the conservation laws in the continuum theory, such as the mass conservation equation,

$$\frac{\partial \rho_L (1 - C)}{\partial t} + \nabla \cdot [(1 - C) \rho_L \mathbf{u}_L] = 0, \qquad [11]$$

and the momentum conservation equation,

$$\frac{\partial}{\partial t} [\rho_L (1 - C) \mathbf{u}_L] + \nabla \cdot [(1 - C) \rho_L \mathbf{u}_L \mathbf{u}_L] = -\nabla [(1 - C) P_L]$$

$$+ \nabla \cdot [(1 - C)(\tau_L + \tau_L^T)] + (1 - C) \rho_L \mathbf{g} + \mathbf{M}_L. \qquad [12]$$

Here the subscript "L" represents the liquid phase, $\mathbf{u}_L$ is the mean fluid velocity, ρ_L is the fluid density, P_L is the pressure stress, τ_L is the mean viscous stress, $\mathbf{M}_L = -\mathbf{M}_P$ represents the momentum transmission between the phases, or, the interaction force between the phases and τ_L^T is the fluctuating stress tensor of the liquid phase, which is expressed as

$$\tau_L^T = -\rho_L \widetilde{\mathbf{u}_L' \mathbf{u}_L'}. \qquad [13]$$

Here the symbol " $\sim$ " represents the volumetric mean value.

The liquid phase in two-phase flow is often considered as incompressible, i.e. ρ_L is a constant. In such a case, solving the problems of the motion of solid/liquid two-phase flow means finding solutions of the series equations mentioned above under some given initial and boundary conditions. Owing to the complex nature of the problems, it is only possible to obtain explicit solutions for some special cases, which is what we try to do below.

3. THE PARTICLE VELOCITY DISTRIBUTION FUNCTION FOR TWO-PHASE FLOW

For dilute steady flow conditions, the effect of the collisions represented by $(\partial f/\partial t)_c$ in [1] is so small that it can be neglected without obvious inaccuracies, and thus the Boltzmann equation is simplified to

$$\mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{F}f) = 0. \tag{14}$$

In this case, by analogy with the procedures in the kinetic theory of gas molecules we can easily obtain the particle velocity distribution function with the variational method. According to Wang & Ni (1990), an H-function for the particles may be defined in a similar manner to that used in the kinetic theory of gases:

$$H = \int f \ln f \, d\mathbf{v}, \tag{15}$$

in which $d\mathbf{v} = dv_1 \, dv_2 \, dv_3$. For the equilibrium state, a minimum H is expected. Thus, a variation equation is given as

$$\delta H = \delta \int f \ln f \, d\mathbf{v} = 0, \tag{16}$$

whose additional conditions consist of: the conservation condition of the particle number per unit volume,

$$\delta n = \delta \int f \, d\mathbf{v} = 0; \tag{17}$$

the conservation of momentum,

$$\delta \int m\mathbf{v} f \, d\mathbf{v} = 0; \tag{18}$$

and the conservation of kinetic energy,

$$\delta \int \tfrac{1}{2} m v^2 f \, d\mathbf{v} = 0. \tag{19}$$

With [17] multiplied by a Lagrange operator $\lambda' (= \lambda + 1)$ and the scalar product of [18] and the vector $(1/m)\mathbf{b}$, which is independent of $\mathbf{v}$, and also with [19] multiplied by another operator β, the variational result becomes

$$\begin{aligned} f &= \exp(\lambda + \mathbf{b} \cdot \mathbf{v} - \tfrac{1}{2} m\beta v^2) \\ &= \exp(\lambda + b_i v_i - \tfrac{1}{2} m\beta v^2) \\ &= A \exp(\lambda - \tfrac{1}{2} m\beta V^2), \end{aligned} \tag{20}$$

where $V^2 = \Sigma(v_i - v_{0i})^2$, the square value of the particle peculiar velocity, $v_{0i} = b_i/m\beta$, is the reference characteristic velocity of the particles, $A = \exp[\tfrac{1}{2} m\beta(v_{01}^2 + v_{02}^2 + v_{03}^2)]$ and λ is defined as

$$\lambda = \int m\beta \mathbf{F} \cdot d\mathbf{x} = m\beta \int F_i \, dx_i, \tag{21}$$

from [20] and [14]. Therefore, five unknown parameters, i.e. A, $v_{0i}(i = 1, 2, 3)$ and β, remain to be determined in [20].

For a simple two-dimensional flow, the comprehensive unit mass action force on a particle in either the x- or y-direction is expressed in the following form:

$$F_x = \frac{3}{4} \frac{C_d}{D} \frac{\rho_L}{\rho_P} |\mathbf{v}_L - \mathbf{v}|(v_{L1} - v_1) \tag{22}$$

$$F_y = \frac{3}{4} \frac{C_l}{D} \frac{\rho_L}{\rho_d} (v_{L1} - v_1)^2 - \left(1 - \frac{\rho_L}{\rho_P}\right) g. \tag{23}$$

Here C_d and C_l are comprehensive drag coefficients corresponding to F_x and F_y, respectively; D is the particle diameter; ρ_P is the density of the particles and ρ_L is the density of the fluid; v is the particle stochastic velocity and v_1 is its component in the x-direction; $\mathbf{v}_L$ represents the fluid stochastic velocity and v_{L1} is the x-direction component of $\mathbf{v}_L$.

In order to give an explicit solution for the velocity distribution function of the particles, we have to make some approximations. For example, the stochastic velocity $\mathbf{v}_L$ should be replaced by the mean velocity $\mathbf{u}_L$. Note that the comprehensive force on a particle principally consists of the drag force and the pressure gradient force in opposite directions, the statistically averaged F_x should be zero under two-dimensional flow conditions. As an initial approximation of two-dimensional dilute steady shear flows, the motion of the particles, on average, can be taken as in either the steady or equilibrium state. In this case, the virtual mass force and Basset force are relatively small. For the sake of simplicity, these forces can be neglected. If we make another assumption, that $(u_L - v_1)^2$ is irrelevant to the coordinate y, then we have

$$\int F_x\, dx = C_1 \qquad\qquad [24]$$

and

$$\int F_y\, dy = \tfrac{3}{4} C_l \frac{\rho_L}{\rho_P} (u_L - v_1)^2 \frac{y}{D} - \left(1 - \frac{\rho_L}{\rho_P}\right) gy + C_2. \qquad [25]$$

In which, C_1 and C_2 are two integral constants which are determined according to the initial and boundary conditions. Finally, the particle velocity distribution function is obtained by combining the six relations [20]–[25], i.e.

$$f = A \exp\left\{ m\beta \left[\tfrac{3}{4} C_l \frac{\rho_L}{\rho_P} (u_L - v_1)^2 \frac{y}{D} - \left(1 - \frac{\rho_L}{\rho_P}\right) gy + C_1 + C_2 \right] - \tfrac{1}{2} m\beta \sum (v_i - v_{i0})^2 \right\}. \qquad [26]$$

4. COMPARISONS BETWEEN THE THEORIES AND EXPERIMENTAL RESULTS

The experiments were performed in a $4 \times 6\,\text{cm}^2$ sized square pipe system by means of laser Doppler anemometry (LDA) so as to test the theories presented in this paper. In the horizontally equipped square pipe with transparent side walls, clear water is first injected from the intake and the LDA is fixed at a distance of 5 m from it at the same level as the horizontal pipe. Particles also enter through the intake and then circulate with the water with the help of a self-designed jet pump in the end. The range of Reynolds number is $\text{Re} = (8.3–18) \times 10^3$ for all the experiments. Using this apparatus a lot of measured data on the particle and fluid mean velocity distribution, the velocity probability density distribution and the particle concentration distribution were obtained, only some of them are used here due to the limited length of the paper; some characteristics relating to the experiments are given in table 1.

The measured results show that the fluid mean velocity distribution in dilute two-phase flows can be well-described by the revised formula of single liquid phase flow presented by Dou (1987):

$$\frac{u_L}{u_*} = 2.5 \ln\left[1 + \left(\frac{u_* y}{5v}\right) \right] + 7.05 \left[\frac{\dfrac{u_* y}{v}}{1 + \dfrac{u_* y}{v}} \right]^2 + 2.5 \left[\frac{\dfrac{u_* y}{v}}{1 + \dfrac{u_* y}{v}} \right] + 0.5 \left[1 - \cos\left(\pi \frac{y}{H}\right) \right]. \qquad [27]$$

This implies that the influence of the particle concentration on the fluid characteristics is very low for dilute flows. Here H is the half depth of the square pipe, v is the kinetic viscosity coefficient and u_* is the shear velocity.

Now we return to discuss the particle velocity distribution function as shown in [26] with measured data. Noting that the relation between the kinetic energy $\tfrac{1}{2} m_G V_G^2$ of the molecular heat movement, the temperature T and the parameter β is

$$\tfrac{1}{2} m_G V_G^2 = \tfrac{3}{2} kT = \frac{3}{2\beta} \qquad\qquad [28]$$

Table 1. Characteristics of fluid and particles

Run No.	Particle diameter, D (mm)	Particle density, ρ (g/cm³)	Settling velocity, ω (cm/s)	Shear velocity, u_* (cm/s)	Concentration, C (%)
472	1.10	1.60	6.15	6.95	0.06
473	1.10	1.60	6.15	7.41	0.11
474	1.10	1.60	6.15	7.23	0.24
542	0.85	1.60	4.51	4.86	0.41
543	0.85	1.60	4.51	4.34	0.46
561	0.85	1.60	4.51	6.22	0.12
562	0.85	1.60	4.51	6.62	0.17
563	0.85	1.60	4.51	6.33	0.19
564	0.85	1.60	4.51	6.25	0.38
571	0.85	1.60	4.51	7.16	0.04
572	0.85	1.60	4.51	7.34	0.14
573	0.85	1.60	4.51	7.18	0.16
631	0.60	1.60	3.79	3.38	0.01
632	0.60	1.60	3.79	3.57	0.01
633	0.60	1.60	3.79	3.37	0.08
642	0.60	1.60	3.79	4.90	0.06
643	0.60	1.60	3.79	4.95	0.14
661	0.60	1.60	3.79	6.23	0.04
662	0.60	1.60	3.79	6.21	0.05

in the kinetic theory of gases (here k is the Boltzmann constant), and that the dimensions of the parameter $1/m\beta$ should be the square of a velocity, we can assume a relation for the turbulent shear flow in which no heat movement exists. Generally speaking, the analogy between V_G^2 and V_P^2 is a reasonable one. Thus, noting the same orders of magnitude for V_P^2 and the shear velocity u_* in dilute solid/liquid two-phase shear flows along the vertical (Ni *et al.* 1990), a good approximation can be made for the sake of simplicity, or

$$m\beta = \frac{\alpha}{u_*^2},$$ [29]

in which α is a dimensionless coefficient. In addition, theoretically the range of particle stochastic velocity is $(-\infty, +\infty)$ and the velocity distribution function will be reduced to zero as $v \to \pm\infty$. Taking into consideration all previous relations, [26] can be expressed as

$$f = A_1 \exp\left\{-\frac{\alpha}{u_*^2}\left[\frac{3}{4}\frac{\rho_L}{\rho_d}\frac{H}{D}(C_d - C_l\eta)(u_L - v_1)^2 + \frac{1}{2}\sum(v_i - v_{0i})^2\right]\right\},$$ [30]
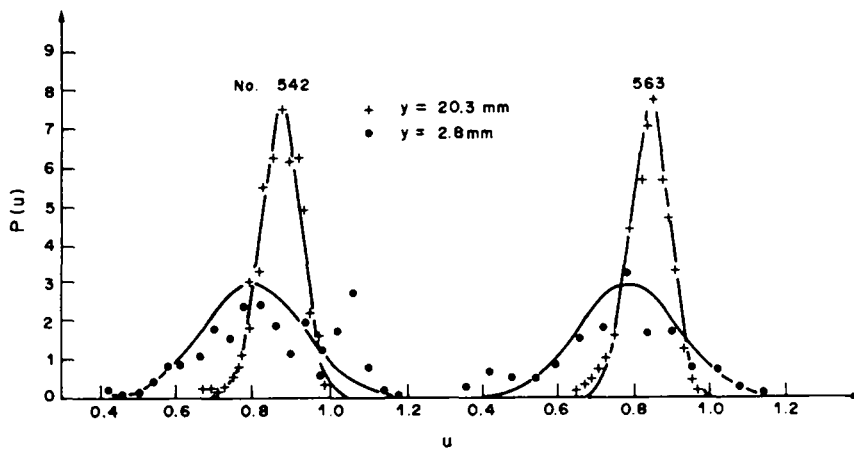

Figure 1. Comparisons between the measured and calculated probability density distributions of particle velocity (I).

in which

$$A_1 = A \exp\left[-\left(1 - \frac{\rho_L}{\rho_d}\right)gy \right] \qquad [31]$$

and $\eta = y/H$, $C_d = 0.4$ and $C_l = 0.2\,C_d$. Moreover, three components v_1, v_2 and v_3 are included in [30], but the mean flow exists only in the x-direction for two-dimensional flows. Thus, finding the integrals for the whole space of v_2 and v_3 successively, we obtain

$$f = A_2 \exp\left\{ -\frac{\alpha}{u_*^2}\left[\frac{3}{4}\frac{\rho_L}{\rho_P}\frac{H}{D}C_d\left(1 - \frac{C_l}{C_d}\eta\right)(u_L - v_1)^2 + \tfrac{1}{2}(v_1 - v_{01})^2\right]\right\}. \qquad [32]$$

Here $A_2 = (2\pi/m\beta)A_1$, which can be determined from the particle numbers per unit volume, as defined in [2], and [32] thus becomes

$$f = n \exp\left\{ -\frac{\alpha}{2}\left(\frac{u_L}{u_*}\right)^2\left[\frac{3}{2}\frac{\rho_L}{\rho_P}\frac{H}{D}C_d\left(1 - \frac{C_l}{C_d}\eta\right)(1 - u)^2 + (u - u_0)^2\right]\right\}$$

$$\div \int_{-\infty}^{+\infty} \exp\left\{ -\frac{\alpha}{2}\left(\frac{u_L}{u_*}\right)^2\left[\frac{3}{2}\frac{\rho_L}{\rho_P}\frac{H}{D}C_d\left(1 - \frac{C_l}{C_d}\eta\right)(1 - u)^2 + (u - u_0)^2\right]\right\}du, \qquad [33]$$

in which $u = v_1/u_L$ and $u_0 = v_{01}/u_L$. The parameters α and u_0 must be determined from the measured data and they are expressed as

$$\alpha = 0.2\left(1 + 0.22\frac{\sqrt{gD}}{u_*}\right)\frac{\eta}{1 + \eta} \qquad [34]$$

$$u_0 = 0.3\left(0.45 + 0.1\frac{\sqrt{gH}}{u_*}\right)\frac{1 + \eta}{1 + 1.6\sqrt{5\dfrac{\omega}{\sqrt{gH}}}}. \qquad [35]$$

Here ω is the settling velocity of a particle. In figures 1 and 2, the comparisons between the probability density distribution and the theoretical results of $P(u) = (1/n)f$ are given at two points for each vertical, i.e. at $y = 2.8$ and $y = 20.3$ mm. We can see that good agreement between the theoretical and experimental results has been achieved.

With a given particle velocity distribution function, the particle mean velocity u_P can be found easily by the relation

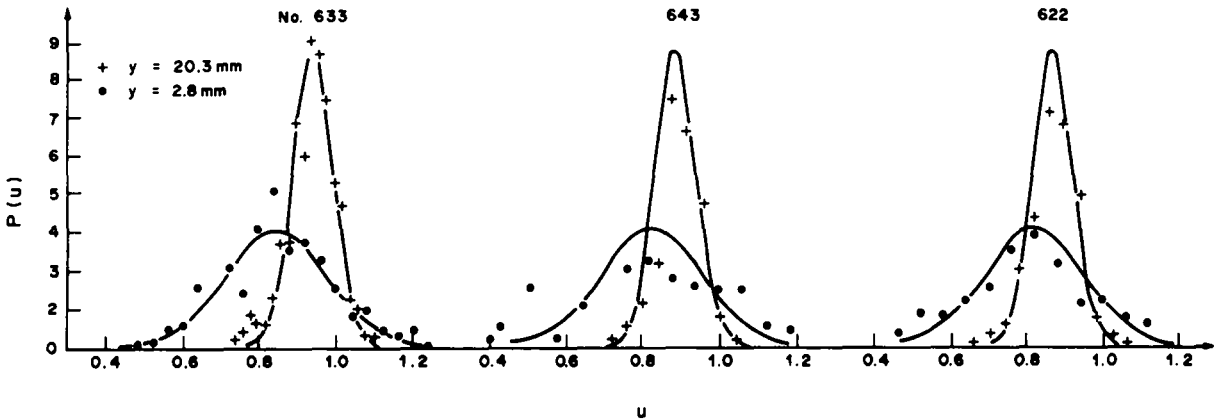$$\frac{u_P}{u_L} = \frac{1}{n}\int_{-\infty}^{+\infty} uf\,du. \qquad [36]$$

Figure 2. Comparisons between the measured and calculated probability density distributions of particle velocity (II).
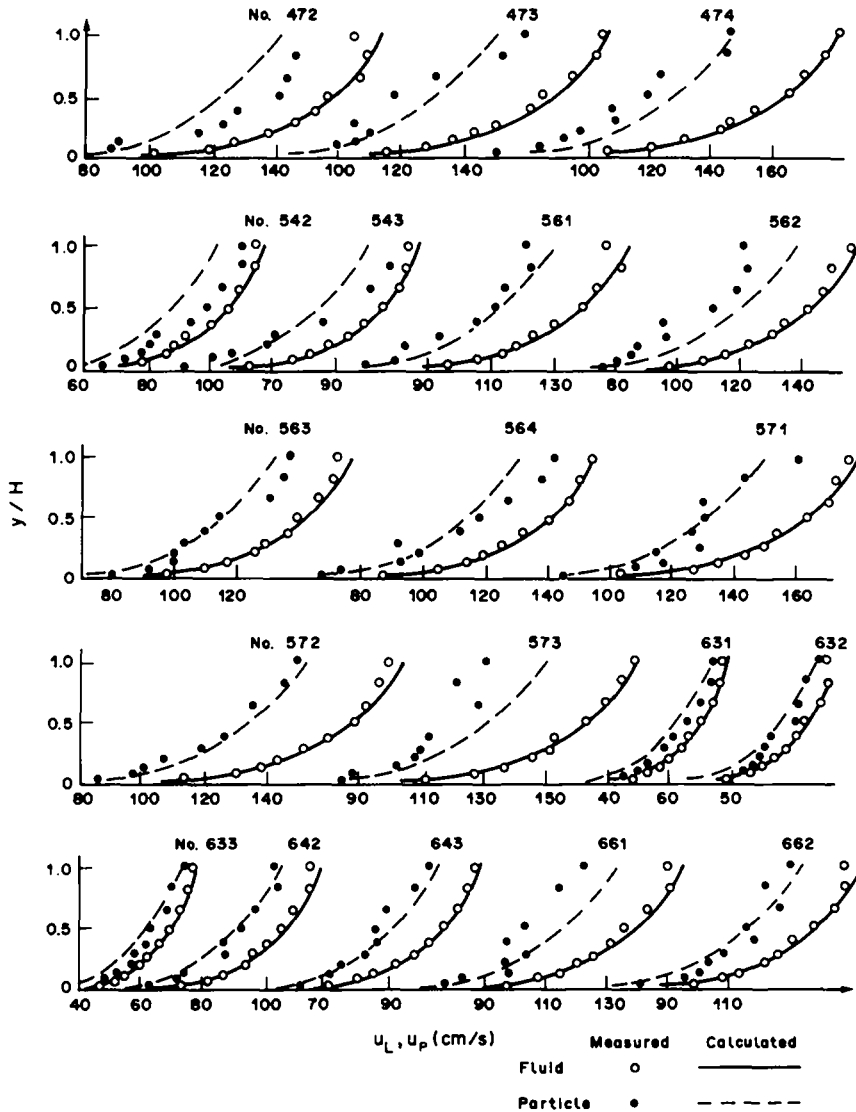
Figure 3. Test of [27] and [36].

A test of [27] and [36] with the measured data is shown in figure 3 and satisfactory agreements are also achieved for both the liquid phase (—) and the solid phase (· ·).

5. CONCLUSION

Both theories and experiments on solid/liquid two-phase flow have been discussed, and the main conclusions drawn are as follows:

(1) For dilute two-phase flows, the solid phase may be described by the kinetic theory and the liquid phase still be described by the continuum theory.

(2) By analogizing the treatments in the kinetic theory of gas molecules for the equilibrium state, the particle velocity distribution function can be derived for steady two-dimensional flows.

(3) Once the particle velocity distribution function is given, the particle mean velocity distribution is readily calculated from [36].

(4) All relations given in this paper agree very well with the measured data by means of LDA.

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